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# A certain double-well potential related to SU(2) symmetry

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**Abstract.** In the present paper we analyse a class of partially exactly solvable one-dimensional double-well potentials. We show how the underlying SU(2) dynamical symmetry makes it possible analytically to find several energy levels.

## 1. Introduction

The importance and usefulness of the double-well potential

$$U(x) = -Ax^2 + Bx^4 \tag{1}$$

in quantum mechanics, statistical physics or field theory can hardly be overestimated. Nevertheless, there are no exact, analytical, results in the problems involving the potential (1). On the other hand, it was reported recently [1–3] that some exact eigenvalues of the double-well potential

$$V(x) = V_0(A\cosh ax - 1)^2 \qquad (0 < A < 1) \tag{2}$$

might be found for a certain choice of the parameters $V_0$ and $A$. In the present paper we make use of some general ideas submitted in [2] and [1] to study the properties of the model (2) and the reasons for its exact solvability. Usually, one expects that exact solvability of the spectrum indicates the presence of a geometrical or dynamical ('hidden') symmetry [4–10].

Several examples of solvable potentials generated by the su(1,1) algebra are discussed in [11]. It was also shown that exactly solvable one-dimensional models can be obtained as projections from two-dimensional models which have a symmetry. The relation of such one-dimensional models (Poschl-Teller, Morse, Coulomb and other potentials) to the su(2) algebra was discussed by Alhassid *et al* [12] and by Ghosh *et al* [13]. There are two characteristic features common to all of these exactly solvable one-dimensional models:

(i) all the bound states are known exactly;

(ii) one-dimensional models are projections of two-dimensional models.

In the present paper we will show how the relation with the su(2) algebra underlies the solvability of the potential (2). In this sense, the model is similar to the above mentioned exactly solvable one-dimensional models related to the SU(2) dynamical symmetry. However, it differs from them in both listed points:

(i) only a finite sequence of the lowest-lying levels is known;

(ii) the exact solvability is an inherent feature of the model (2) without any relations to two-dimensional models.

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We will also propose some other related exactly solvable models, which are part of the very wide class of models, as discussed in [14].

The Schrödinger equation for a particle of mass $m$ in the potential (2) is

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_0(A\cosh ax - 1)^2 - E\right]\Psi(x) = 0. \tag{3}$$

$V_0$ and $A$ can be expressed in terms of dimensionless parameters $S$ and $B$:

$$V_0 = \frac{\hbar^2 a^2}{8m}(2S+1)^2 \tag{4}$$

$$A = \frac{B}{(2S+1)} \tag{5}$$

$$-\tfrac{1}{2} < S < \infty \qquad B > 0. \tag{6}$$

Furthermore, let us introduce the dimensionless unit $\zeta = ax$ to obtain the following equation

$$[H - \epsilon]\,\Psi(\zeta) = 0 \tag{7}$$

where

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} + \frac{1}{4}B^2\sinh^2\zeta - B\left(S + \frac{1}{2}\right)\cosh\zeta \tag{8}$$

$$\epsilon = \frac{2m}{\hbar^2 a^2}E - \frac{1}{4}(2S+1)^2\left[1 + \left(\frac{B}{2S+1}\right)^2\right]. \tag{9}$$

The analogy between the Hamiltonian $H$ and the spin Hamiltonian

$$H_s = -S_z^2 - BS_x \tag{10}$$

was underlined in [2]. It is possible to exploit this formal analogy in order to find the reasons why some energy levels and eigenfunctions can be given in a closed analytical form. It can be shown that SU(2) is the dynamical symmetry group of the Hamiltonian (8). The well known facts concerning the representations of the su(2) Lie algebra allow us to create a well defined procedure to construct the analytically obtainable levels. Some further conclusions can be drawn on this basis as well.

## 2. A certain representation of the su(2) Lie algebra

The following operators acting on a subset of $L^2(\mathrm{R})$ can be defined [2]:

$$S_x = S\cosh\zeta - \frac{B}{2}\sinh^2\zeta - \sinh\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta} \tag{11}$$

$$S_y = \mathrm{i}\left\{-S\sinh\zeta + \frac{B}{2}\sinh\zeta\cosh\zeta + \cosh\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\right\} \tag{12}$$

$$S_z = \frac{B}{2}\sinh\zeta + \frac{\mathrm{d}}{\mathrm{d}\zeta}. \tag{13}$$

These operators fulfil the commutation relations of spin operators, that is

$$[S_i, S_j] = \mathrm{i}\epsilon_{ijk}S_k \tag{14}$$

which means that they are the standard basis of a representation of the su(2) algebra, multiplied by i. The special feature of the representation (11)–(12) is that

$$S_x^2 + S_y^2 + S_z^2 \equiv S(S+1)\cdot\mathrm{Id}. \tag{15}$$

The Hamiltonian constructed according to (10) using the representation defined above is (8), therefore one can express the double-well Hamiltonian (8) in terms of elements of the su(2) Lie algebra. The su(2) algebra describes the dynamical symmetry of the system, i.e. any subspace of $L^2(\mathrm{R})$ invariant under the algebra is obviously invariant under the action of the Hamiltonian as well. The existence of a finite-dimensional invariant subspace allows to replace the original problem (7) by a matrix problem (i.e. to restrict the problem to a *finite-dimensional subspace*).

Let us define the raising and lowering operators [15, section 7.3, 16] which, in terms of the spin operators, are expressed by

$$S_+ = S_x + \mathrm{i}S_y \tag{16}$$

$$S_- = S_x - \mathrm{i}S_y. \tag{17}$$

It seems worth investigating whether a finite-dimensional invariant subspace can exist and what dimension it can have. To this end we follow the discussion in [15] and consider an invariant finite-dimensional and irreducible subspace $V$ of the representation space of the algebra. Using the fact that $S_z$ must have at least one eigenvector in this space (as it is a complex space) and employing the 'raising' and 'lowering' feature of $S_+$ and $S_-$ respectively, we can construct a set of non-zero vectors $f_l \ldots f_u$ belonging to different *eigenvalues* $l, l+1, \ldots, u$ *of* $S_z$ *(and therefore linearly independent), such that* $S_+ f_u = 0$ and $S_- f_l = 0$.

The Casimir operator for su(2) in terms of the raising and lowering operators has the form

$$S^2 = S_+S_- + S_z^2 - S_z = S_-S_+ + S_z^2 + S_z. \tag{18}$$

As has already been stated, for the representation considered here it is identically equal to $S(S+1)$. Therefore

$$S(S+1)f_u = S^2 f_u = [S_-S_+ + S_z(S_z+1)]f_u = u(u+1)f_u \tag{19}$$

and

$$S(S+1)f_l = S^2 f_l = [S_+S_- + S_z(S_z-1)]f_l = l(l-1)f_l. \tag{20}$$

Combining (19) and (20) we obtain $u(u+1) = S(S+1) = l(l-1)$. As $u \geqslant l$ by definition, we find that $l = -u$; $u \geqslant 0$. Because $u - l = n$, where $n$ is a natural number, we find that $2u$ must be a non-negative integer. Furthermore, as $S > -1/2$, we have $S = u$. This is an essential conclusion: the existence of a finite-dimensional subspace which is invariant under the algebra generated by the operators $S_x$, $S_y$, $S_z$ requires that $2S$ should be integer. The $2S+1$ vectors $f_k = S_-^{u-k} f_u$; $n = -u, -u+1, \ldots, u$ belong to different eigenvalues of $S_z$ and therefore they are linearly independent. The subspace they span is invariant. Because we assumed that $V$ is irreducible, the subspace spanned by these vectors must be V itself.

Hence, an irreducible finite-dimensional subspace invariant under the operators $S_x$, $S_y$, $S_z$ can exist only when $2S$ is a non-negative integer; its dimension is then $2S+1$ and it is spanned by the basis

$$f_S, S_- f_S = f_{S-1}, \ldots, S_-^{2S} f_S = f_{-S}. \tag{21}$$

## 3. Construction of the invariant subspace and the solution of the eigenvalue problem for $S = 1$

As an example, for simplicity, let us consider the case of $S = 1$. Nevertheless, the cases $S = 3/2$, $S = 2$ can be solved in the same manner.

In the discussed case the invariant subspace is spanned by

$$f_1(\zeta) = \exp(\zeta - \tfrac{1}{2}B\cosh\zeta) \qquad f_0(\zeta) = \sqrt{2}\exp(\tfrac{1}{2}B\cosh\zeta)$$
$$f_{-1}(\zeta) = \exp(-\zeta - \tfrac{1}{2}B\cosh\zeta). \tag{22}$$

The Hamiltonian in this subspace has the form

$$H = -S_z^2 - BS_x = \begin{bmatrix} -1 & -B/\sqrt{2} & 0 \\ -B/\sqrt{2} & 0 & -B/\sqrt{2} \\ 0 & -B/\sqrt{2} & -1 \end{bmatrix}. \tag{23}$$

The eigenvectors are

$$\phi_0 = (1, 0, -1) \qquad \phi_1 = (B, \sqrt{2}\epsilon_2, B) \qquad \phi_2 = (B, \sqrt{2}\epsilon_0, B) \tag{24}$$

which in terms of the basis functions reads

$$\begin{aligned} \Psi_0 &= C_0\left(B\cosh\zeta - \frac{1-\sqrt{1+4B^2}}{2}\right)\exp(-\tfrac{1}{2}B\cosh\zeta) \\ \Psi_1 &= C_1\sinh\zeta\exp(-\tfrac{1}{2}B\cosh\zeta) \\ \Psi_2 &= C_2\left(B\cosh\zeta - \frac{1+\sqrt{1+4B^2}}{2}\right)\exp(-\tfrac{1}{2}B\cosh\zeta). \end{aligned} \tag{25}$$

The corresponding eigenvalues of the matrix Hamiltonian are

$$\begin{aligned} \epsilon_0 &= -\tfrac{1}{2}\left(1+\sqrt{1+4B^2}\right) \\ \epsilon_1 &= -1 \\ \epsilon_2 &= -\tfrac{1}{2}\left(1-\sqrt{1+4B^2}\right). \end{aligned} \tag{26}$$

The three lowest energy levels of the original system with the potential (2) are, therefore, (see (9))

$$\begin{aligned} E_0 &= \frac{\hbar^2a^2}{8m}\left(B^2+7-2\sqrt{1+4B^2}\right) = \frac{9\hbar^2a^2}{8m}\left(A^2+\frac{7}{9}-\frac{2}{9}\sqrt{1+36A^2}\right) \\ E_1 &= \frac{\hbar^2a^2}{8m}\left(B^2+5\right) = \frac{9\hbar^2a^2}{8m}\left(A^2+\frac{5}{9}\right) \\ E_2 &= \frac{\hbar^2a^2}{8m}\left(B^2+7+2\sqrt{1+4B^2}\right) = \frac{9\hbar^2a^2}{8m}\left(A^2+\frac{7}{9}+\frac{2}{9}\sqrt{1+36A^2}\right) \end{aligned} \tag{27}$$

where we have returned to the original parameter $A$, according to (4) and (5).

We can determine how the energy levels lie in relation to the top of the potential hump $V_m = V_0(1-A)^2$. For $S = 1$, $V_0 = 9\hbar^2a^2/8m$, one finds that $E_2$ always lies above the potential hump, $E_2 > V_m$. For $A$ small enough, $A < \frac{2}{9}$, $E_0$ and $E_1$ can both be found within the wells. When $A$ increases, $E_0$ and $E_1$ rise and for $A$ large enough, $A > \frac{2}{5}$, all the energy levels lie above the top of the potential hump.

## 4. The cases of larger $S$ and $2S$ non-integer

For $S > 2$, $2S$ integer, the problem is reduced to a finite matrix problem in the same way as described above. There are, however, no analytical solutions, as matrices of higher dimensions cannot be exactly diagonalized. On the other hand, let us remark that the matrix Hamiltonian we obtain is always a finite three-diagonal matrix of the form

$$H = \begin{bmatrix} a_1 & b_1 & 0 & \dots & & \\ -b_1 & a_2 & b_2 & 0 & \dots \\ 0 & -b_2 & a_3 & b_3 & \dots \\ & & \vdots & & \\ & \dots & 0 & -b_{2S} & a_{2S+1} \end{bmatrix}. \tag{28}$$

From the point of view of numerical computations, the eigenvalue problem for this matrix is of complexity $S$, whereas for a general finite problem it would be of complexity $S^2$. Moreover, solutions of this problem arbitrarily close to exact ones may be achieved without changing the dimension of the matrix, which is not the case for a general eigenvalue problem in an infinite space.

When $2S$ is non-integer, there cannot exist any invariant subspace. However, $S_- f_{-S} = 0$, which makes it possible to create a $\uparrow_u$ representation [15]. The Hamiltonian (8) in this representation has the form

$$H = \begin{bmatrix} a_1 & b_1 & 0 & \dots & & \\ -b_1 & a_2 & b_2 & 0 & \dots \\ 0 & -b_2 & a_3 & b_3 & \dots \\ 0 & 0 & -b_3 & a_4 & \dots \\ & & & & \ddots \end{bmatrix} \tag{29}$$

which can be an appropriate starting point for approximate estimations.

## 5. Other potentials generated by su(2)

For any operators of the form $g(x)+h(x)\frac{\mathrm{d}}{\mathrm{d}x}$ and satisfying the su(2) commutation relations, the most general combination including up to the second power of these operators, satisfying the three conditions characteristic of the Schrödinger equation, i.e.

(i) there is no first derivative term,

(ii) the coefficient at $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ is constant,

(iii) the potential (i.e. the non-derivative term) is real,

leads to several classes of potentials, as analysed in [14]. Two of them are the potentials listed in [2]:

(i) Single- and double-well potentials, given by

$$V_1(\zeta) = \frac{B^2}{4}\left(\sinh\zeta - \frac{C}{B}\right)^2 - B\left(S + \frac{1}{2}\right)\cosh\zeta \tag{30}$$

which become asymmetrical for $C \neq 0$;

(ii) periodical potentials

$$V_2(\zeta) = \frac{B^2}{4}\left(\sin\zeta - \frac{C}{B}\right)^2 - B\left(S + \frac{1}{2}\right)\cos\zeta \tag{31}$$

which for $C \neq 0$ is a potential of the kind of a one-dimensional heteroatomic lattice. The method presented above is fully applicable for this potential and one can find wavefunctions corresponding to $k = 0$ ($k$ is the wavevector) and $k = 1/2$ for $S$ integer and half-integer, respectively [2].

Another example of such a class of potentials is given in [14].

## 6. Conclusions

The reasons for the exact solvability of the Razavy double-well-type potential have been discussed. As the Hamiltonian could be expressed as a combination up to the second power of operators belonging to the su(2) algebra, one can expect the appearance of invariant subspaces. In this case, however, in contrast to many other exactly solvable one-dimensional models, only a finite sequence of infinite number of bound states is known. This feature is related to the fact that the Casimir operator $S_x^2 + S_y^2 + S_z^2 \equiv S(S+1)\mathrm{I}$ is in this case associated with the model parameter $S$. Therefore, only $2S+1$ levels might be found exactly for half-integer $S$. One finds that these are the lowest energy levels.

The model discussed here turns out to be one of a wider class of models that can be obtained in a similar way, with the same characteristic feature: $2S+1$ energy levels are known exactly.

It should be pointed out that this type of exact solvability is an inherent feature of a one-dimensional model.

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